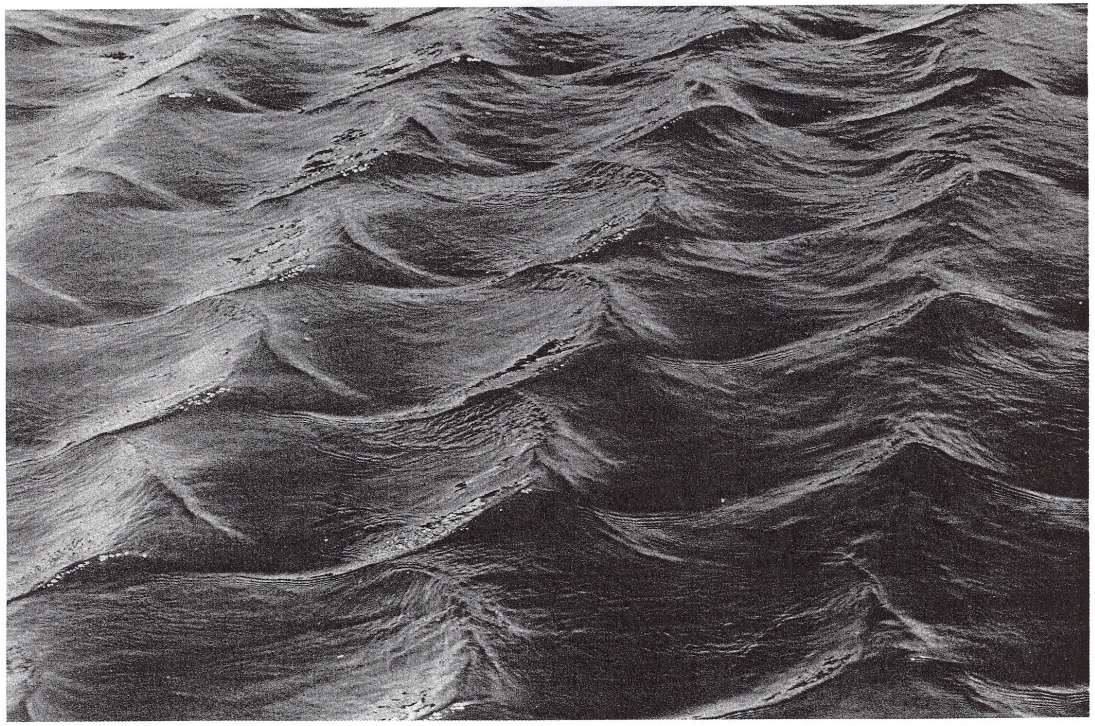
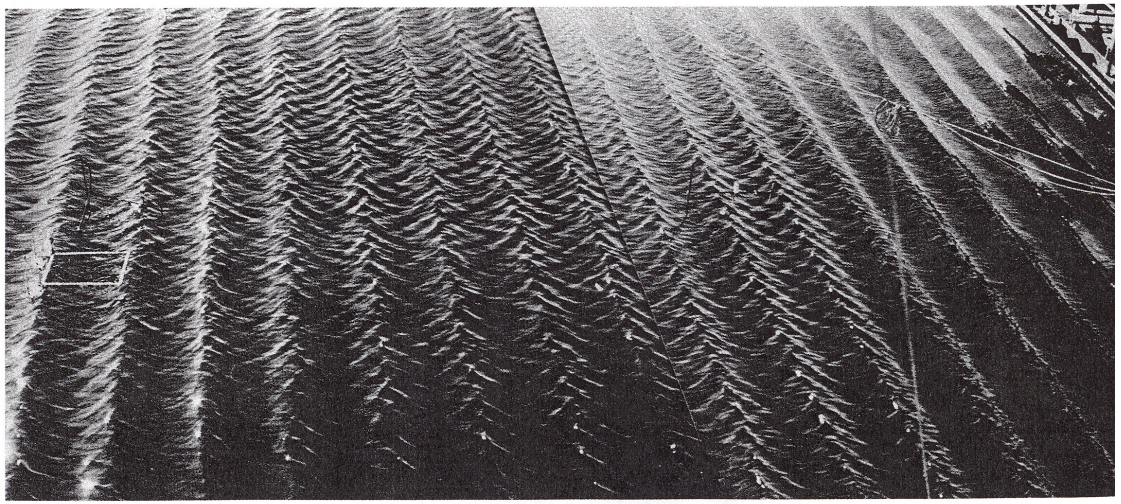
Steady Rotational Water Waves

Walter Strauss

in memory of Saul Abarbanel
ICERM – Aug. 21, 2018



194. Spilling breaking waves. This regular three-dimensional pattern, reminiscent of waves in the open sea, has evolved by nonlinear instability from a uniform train of steep two-dimensional Stokes waves. The waves are propagating from left to right with wavelength 0.75 m. *Photograph by Ming-Yang Su*



195. Overall view of wave evolution. Steep Stokes waves of wavelength 0.75 m are generated in a great outdoor basin of depth 1 m by the oscillating wavemaker at them to evolve into the three-dimensional spilling breaking waves shown above in close-up. *Photograph by Ming-Yang Su*

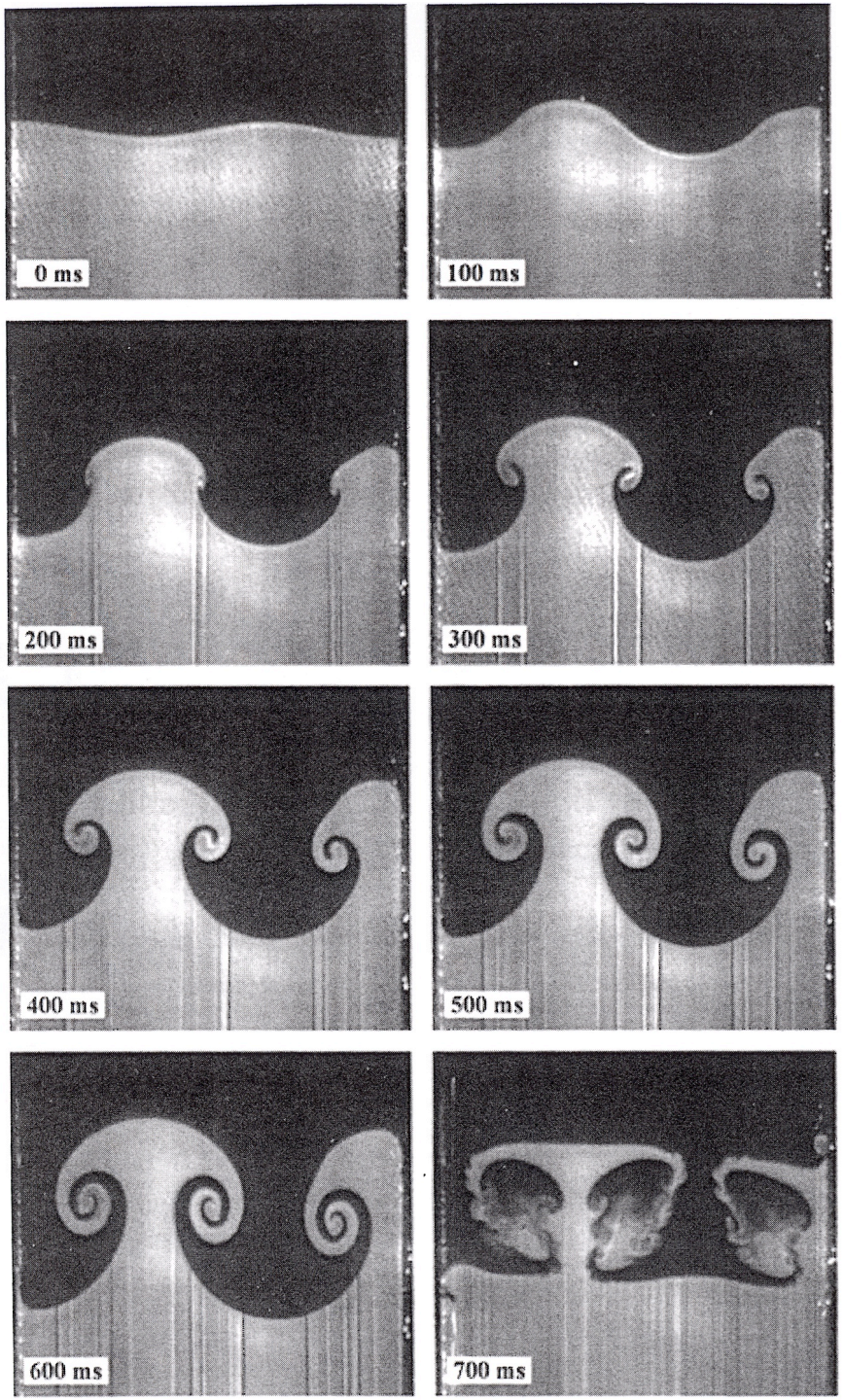


Figure 1

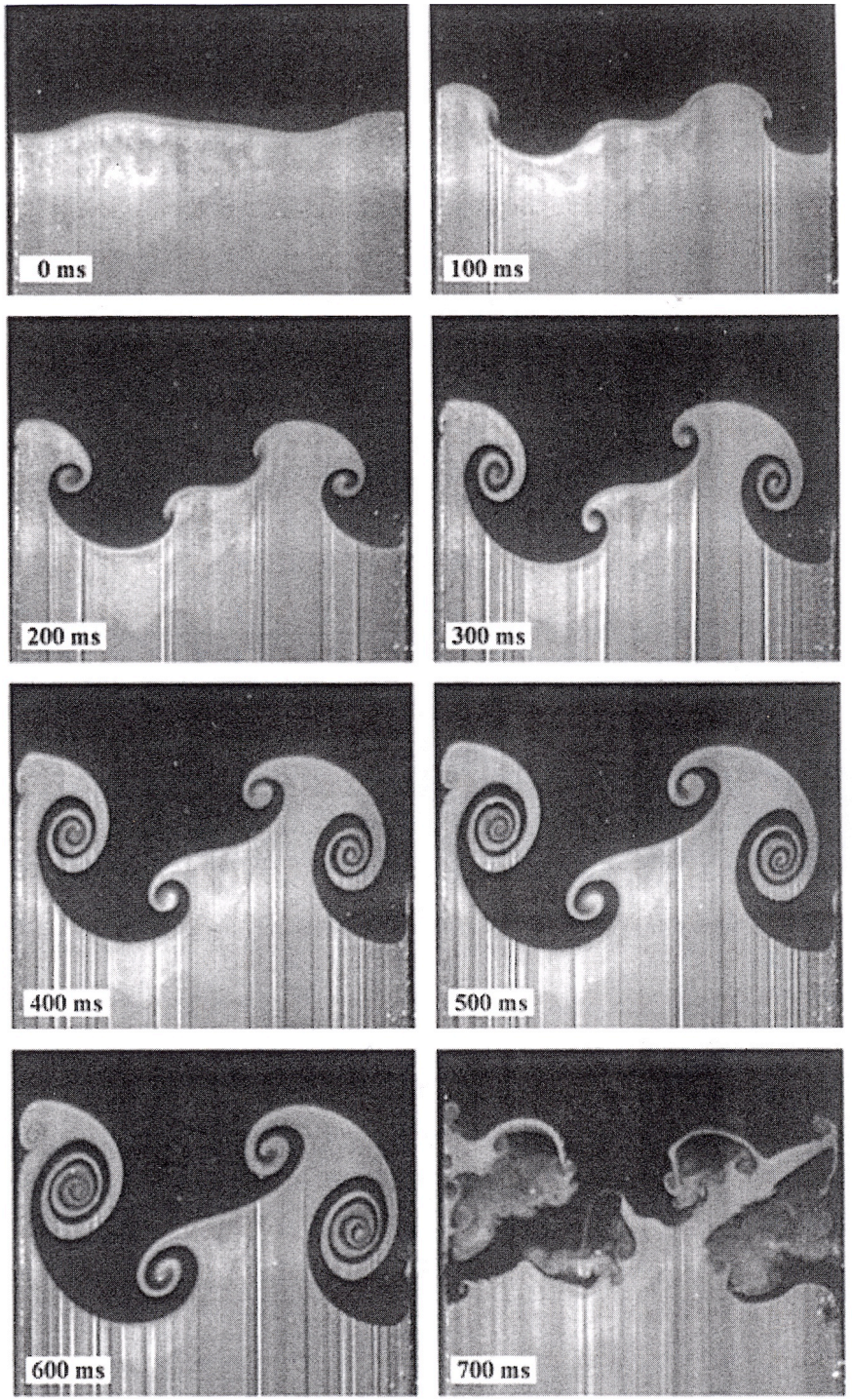
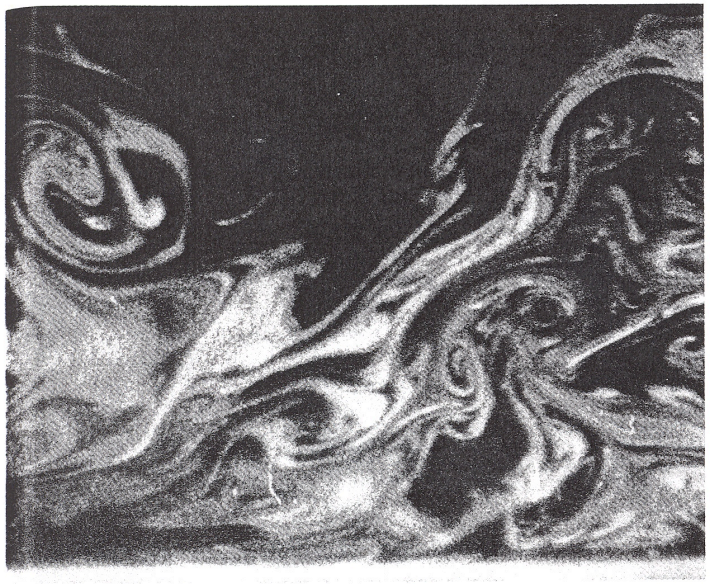
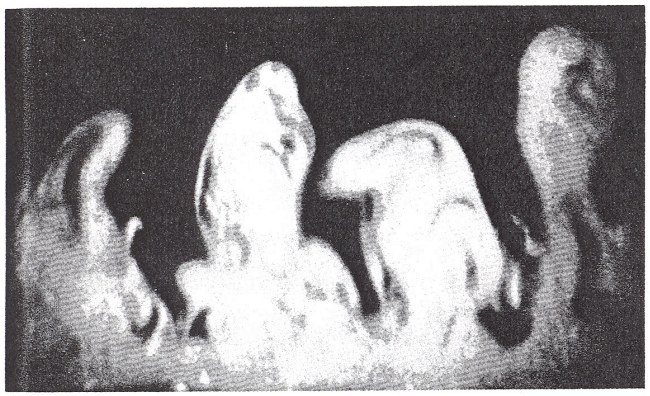
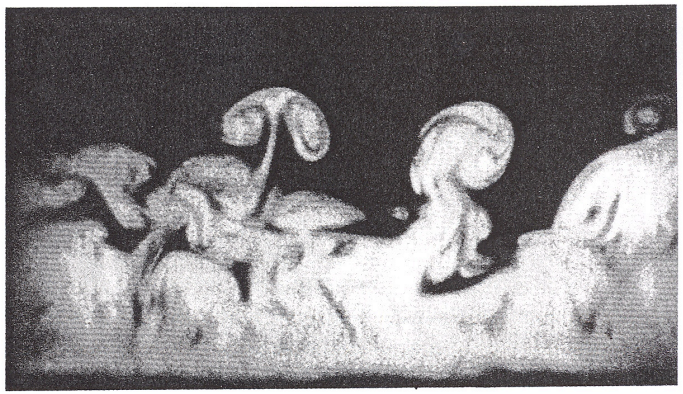


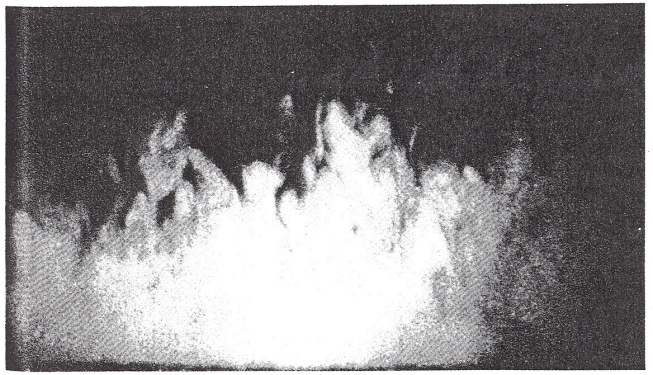
Figure 2

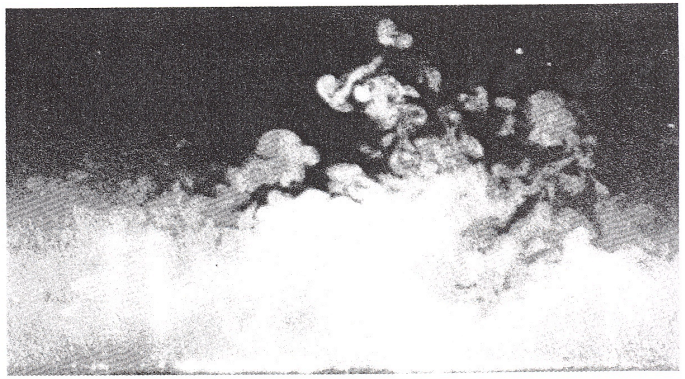


162. "Typical eddy" in a turbulent boundary layer. Oil fog is illuminated by a sheet of laser light to show the lower two-thirds of a turbulent boundary layer in side view. The vortex-ring structure just below and to the right of center, which resembles a sliced mushroom leaning left, is an example of what Falcó has called a "typical eddy." It scales on wall variables (figure 161) rather than on the boundary-layer thickness. *Photograph by R. E. Falco*









History:

- Euler(1750)
- Laplace (1776), Lagrange(1788), Gerstner(1802), Cauchy (1815), Poisson, Airy, Stokes (1847) and many others

Modern analytical theory:

- Approximations for small initial data, shallow water (KdV, NLS, Boussinesq, Kadomtsev-Petviashvili, Davey-Stewartson, etc.)
- Long-time exact solutions for small initial data (Sijue Wu and many others). 3D is easier than 2D.
- Exact steady (traveling) solutions, small and large. 2D is easier than 3D.

This talk

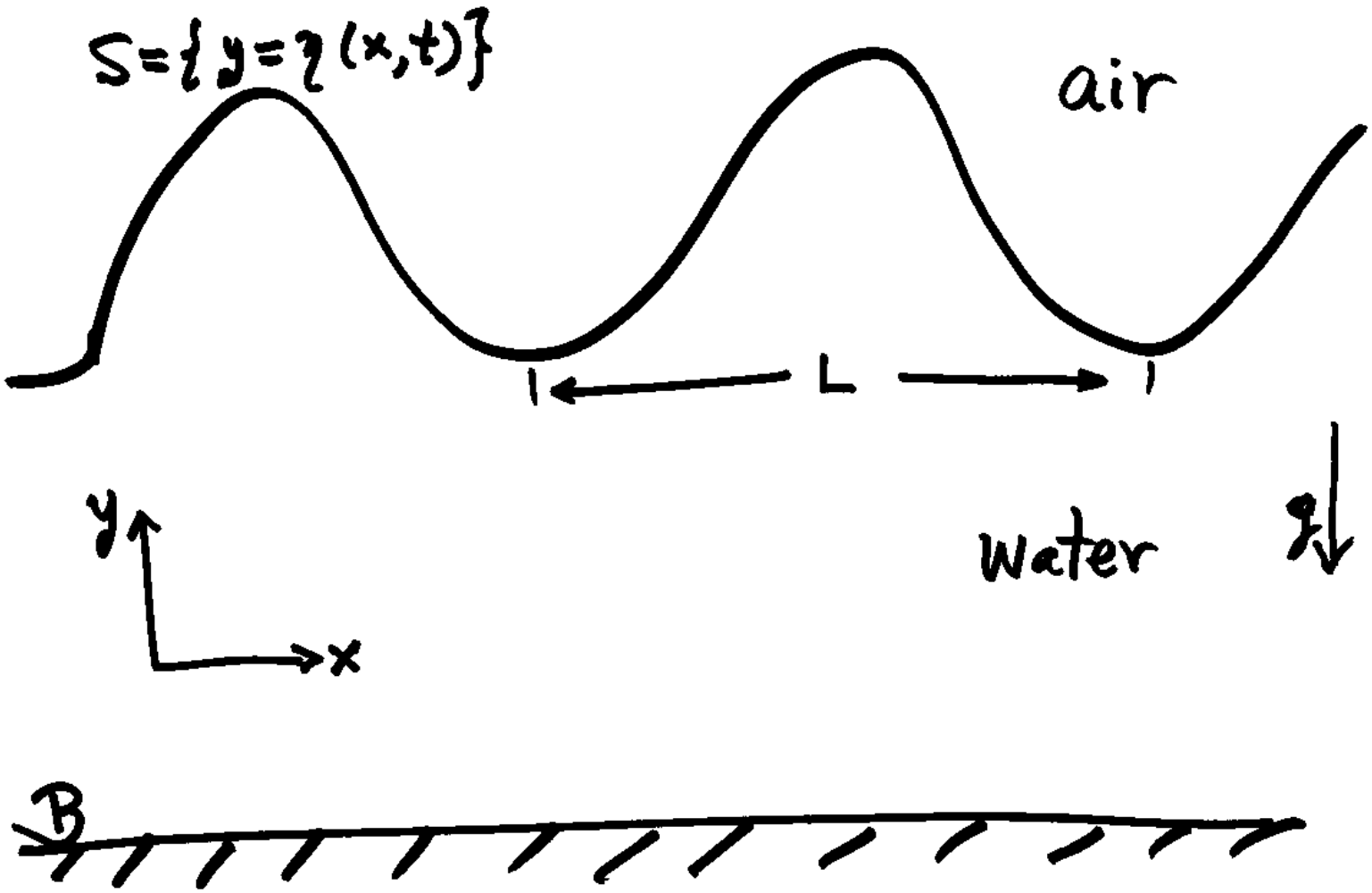
In this talk I'll consider the third theory: steady flow of water in a lake (say) with a flat bottom B and a free surface S under gravity g. Above is air with atmospheric pressure P_{atm}.

The water: 2D, incompressible, inviscid, no surface tension. Even under such severe assumptions the theory is still incredibly rich. There are many exact solutions of the full equations.

What is the effect of vorticity (rotational wave)?

OUTLINE

- The equations
- History
- Periodic waves: existence and simulations
- Waves with stagnation points
- Favorable waves
- Fixing the boundary
- Some ideas for the proofs


S={y=η(x,t)}
air
L
y
x
water
g
B

2D Euler Equations

Velocity $\vec{u} = [u, v] = [u(x, y, t), v(x, y, t)]$, pressure$= P(x, y, t)$, density$= 1$. Inside the fluid [Euler $\sim$1750]:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

On the surface $S = \{y = \eta(x, t)\}$:

$$P = P_{atm}, \quad v = \eta_t + u\,\eta_x$$

On the bottom $B = \{y = -d\}$: $\quad v = 0.$

2D Euler Equations

Velocity $\vec{u} = [u, v] = [u(x, y, t), v(x, y, t)]$, pressure$= P(x, y, t)$, density$= 1$. Inside the fluid [Euler $\sim$1750]:

$$\nabla \cdot \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} + \nabla P = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

On the surface $S = \{y = \eta(x, t)\}$:

$$P = P_{atm}, \quad v = \eta_t + u\,\eta_x$$

On the bottom $B = \{y = -d\}$: $\quad v = 0.$

$$* * * * * * * * * **$$

Define the stream function ψ: $\quad \psi_x = -v, \;\; \psi_y = u$

and the 2D vorticity ω: $\quad \omega = v_x - u_y = -\Delta\psi.$

Streamlines: level curves of ψ.

Steady (i.e. Traveling) Waves

We consider waves of speed c:
velocity $\vec{u}(x - ct, y)$, surface $\eta(x - ct)$.
Therefore we can change to a moving frame: $x - ct \longrightarrow x$.
Then the Euler equations (by eliminating the pressure, we have $\vec{u} \cdot \nabla\omega = 0$) imply that ω and $\psi - cy$ are functionally dependent, so that we get the nonlinear elliptic PDE

$$-\Delta\psi(x, y) = \gamma[\psi(x, y) - cy] \equiv \omega(x, y)$$

where $\Delta\psi = \psi_{xx} + \psi_{yy}$. The vorticity function γ is a completely arbitrary function of one variable; we assume it's single-valued.

Steady (i.e. Traveling) Waves

We consider waves of speed c:
velocity $\vec{u}(x-ct, y)$, surface $\eta(x-ct)$.
Therefore we can change to a moving frame: $x - ct \longrightarrow x$.
Then the Euler equations (by eliminating the pressure, we have $\vec{u} \cdot \nabla\omega = 0$) imply that ω and $\psi - cy$ are functionally dependent, so that we get the nonlinear elliptic PDE

$$-\Delta\psi(x,y) = \gamma[\psi(x,y) - cy] \equiv \omega(x,y)$$

where $\Delta\psi = \psi_{xx} + \psi_{yy}$. The vorticity function γ is a completely arbitrary function of one variable; we assume it's single-valued.

There are two constants:
1. The flux $m = \int_{-d}^{\eta(x)} [u(x,y) - c]\, dy$ is independent of x.
2. The relative stream function $\psi(x,y) - cy$, restricted to the free surface S, is a constant.

We consider waves that are periodic in x.

History in irrotational case

Existence:

Nekrasov, Levi-Civita, Struik (1920's),

Keady & Norbury, Amick & Toland & Fraenkel ($\sim$1980), ...

All of this work strongly uses the fact that ψ is a harmonic function inside the fluid.

Waves with vorticity

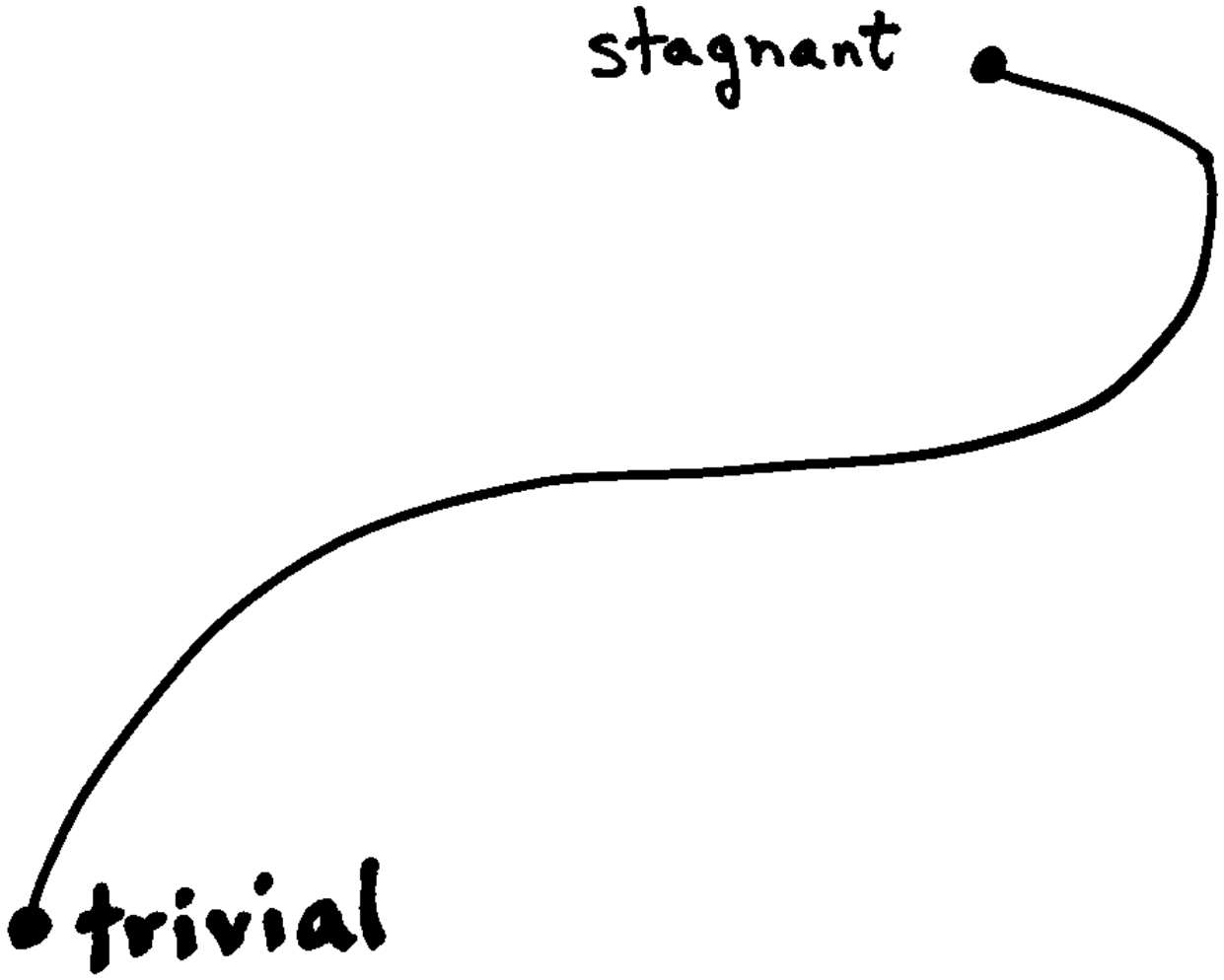
Rotational effects (i.e. due to nontrivial vorticity) are significant for

- wind-driven waves
- waves riding upon a sheared current
- waves near a ship
- tsunamis(!) approaching a shore

Important example: Gerstner (1802)

Existence:
Dubreil-Jacotin (1934) small waves,
Constantin & S (2004) large waves

This situation requires looking inside the fluid because ψ is no longer a harmonic function.


stagnant
trivial

Existence of Traveling Waves

Theorem 1 [Adrian Constantin & S, 2004-2011]
Let arbitrary constants $c > 0, L > 0, p_0 < 0$
and an arbitrary C^∞ function $\gamma(\cdot)$ be given
subject to: either $\omega = \gamma(\cdot) \leq 0$ or $|p_0|$ not too big or L is not too small.

Then there exists a curve $\mathcal{C}$ of C^∞ symmetric traveling waves with $u(x, y) < c$, each one with a single crest and trough per period.

- $\mathcal{C}$ contains a trivial laminar flow with S flat,
- as well as waves for which $\max u \nearrow c$ (“stagnation").

More generally, if the vorticity is a step function, then we get $h \in C^{1+\alpha}, u, v \in C^\alpha, \eta \in C^{1+\alpha}$ for any $0 < \alpha < 1$. $\square$

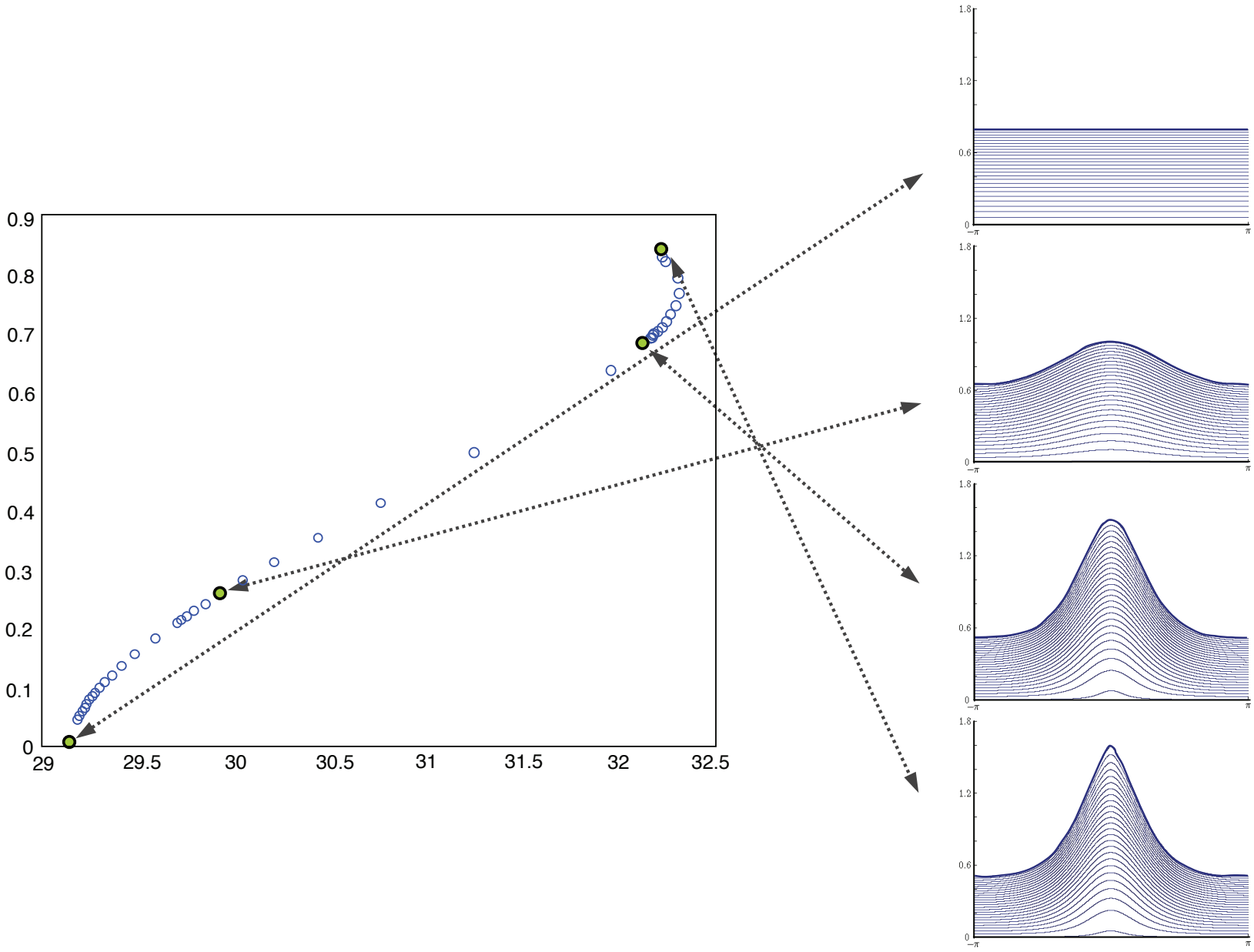
Simulations

[Joy Yueh Ko & S. (2008)]

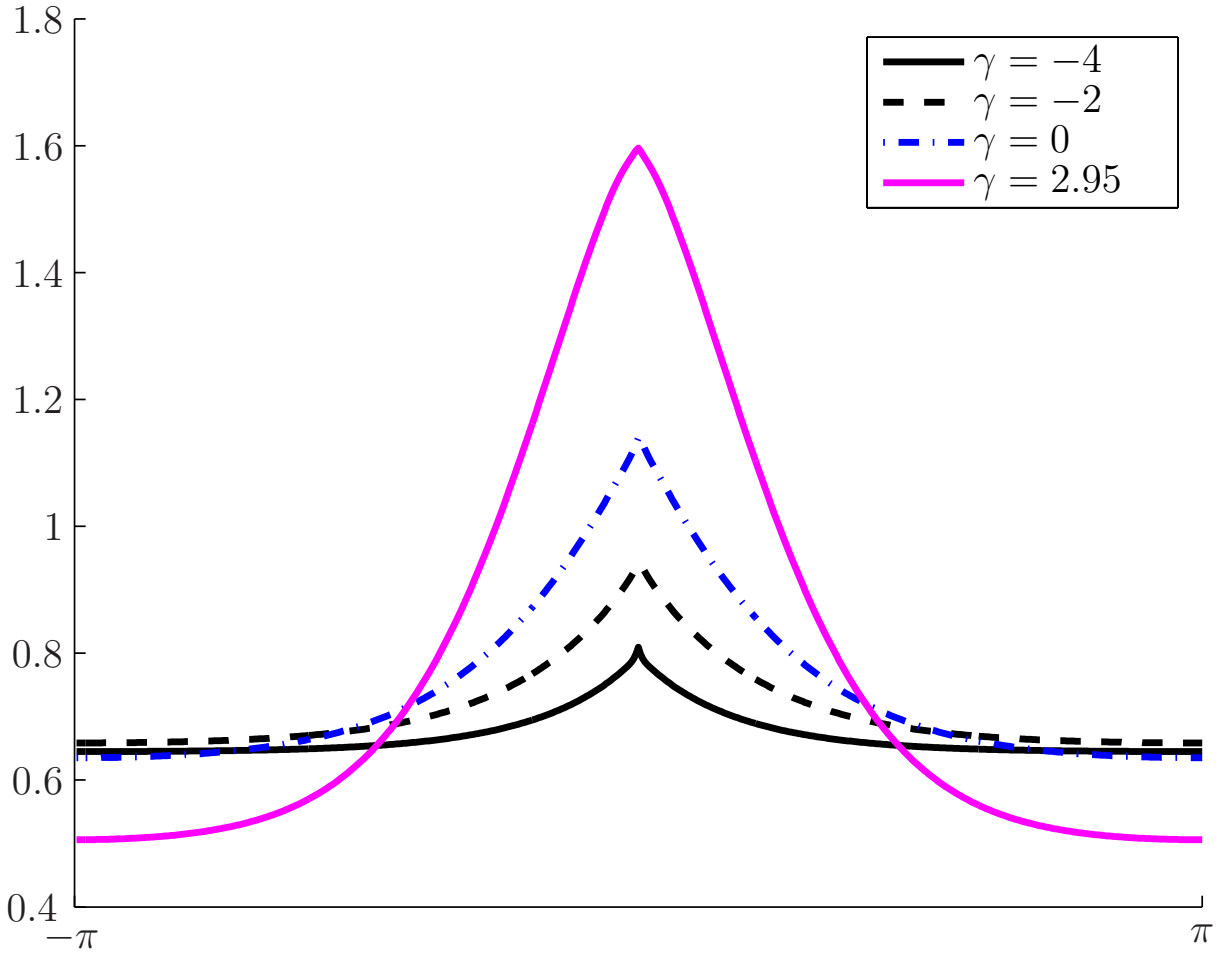
The computations show that the vorticity, even constant vorticity, can have a huge qualitative effect on the solutions!

In the following we compute using numerical continuation.
(We take $L = 2\pi, \;\; p_0 = -2, \;\; g = 9.8.$)

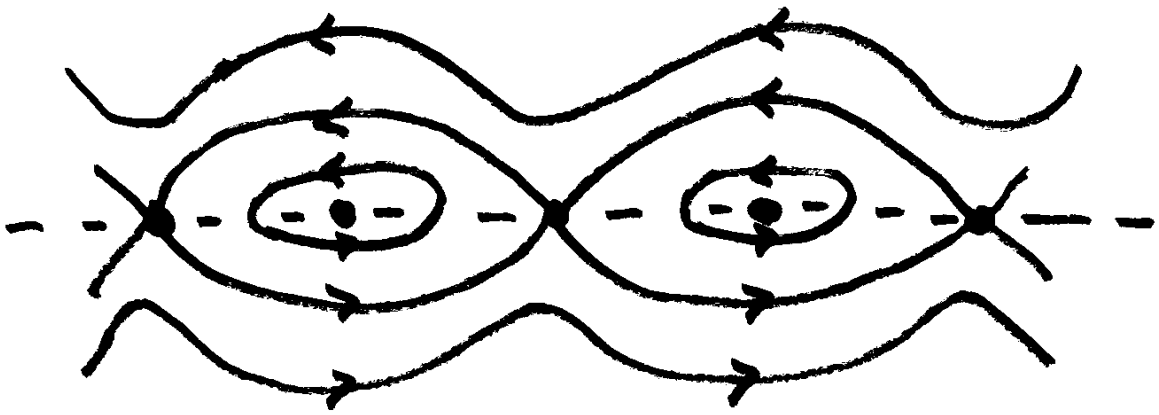
The curve $\mathcal{C}$: amplitude vs. energy for $\gamma \equiv 2.95$ (constant)

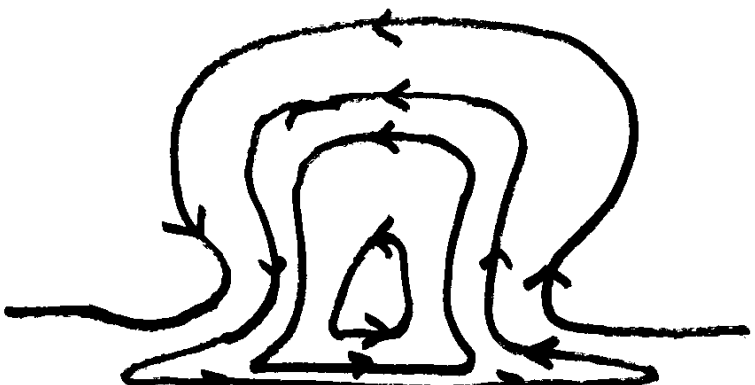


Profiles of waves near stagnation for $\omega \equiv -4,\ -2,\ 0,\ 2.95$



On the other hand, many flows do have stagnation points. Here are two examples.





It is technically easier to compute flows with constant vorticity. Here is a very recent (2018) computation by Dyachenko and Hur.

(This computation considers the case of "adverse" vorticity ω positive and not too small.)

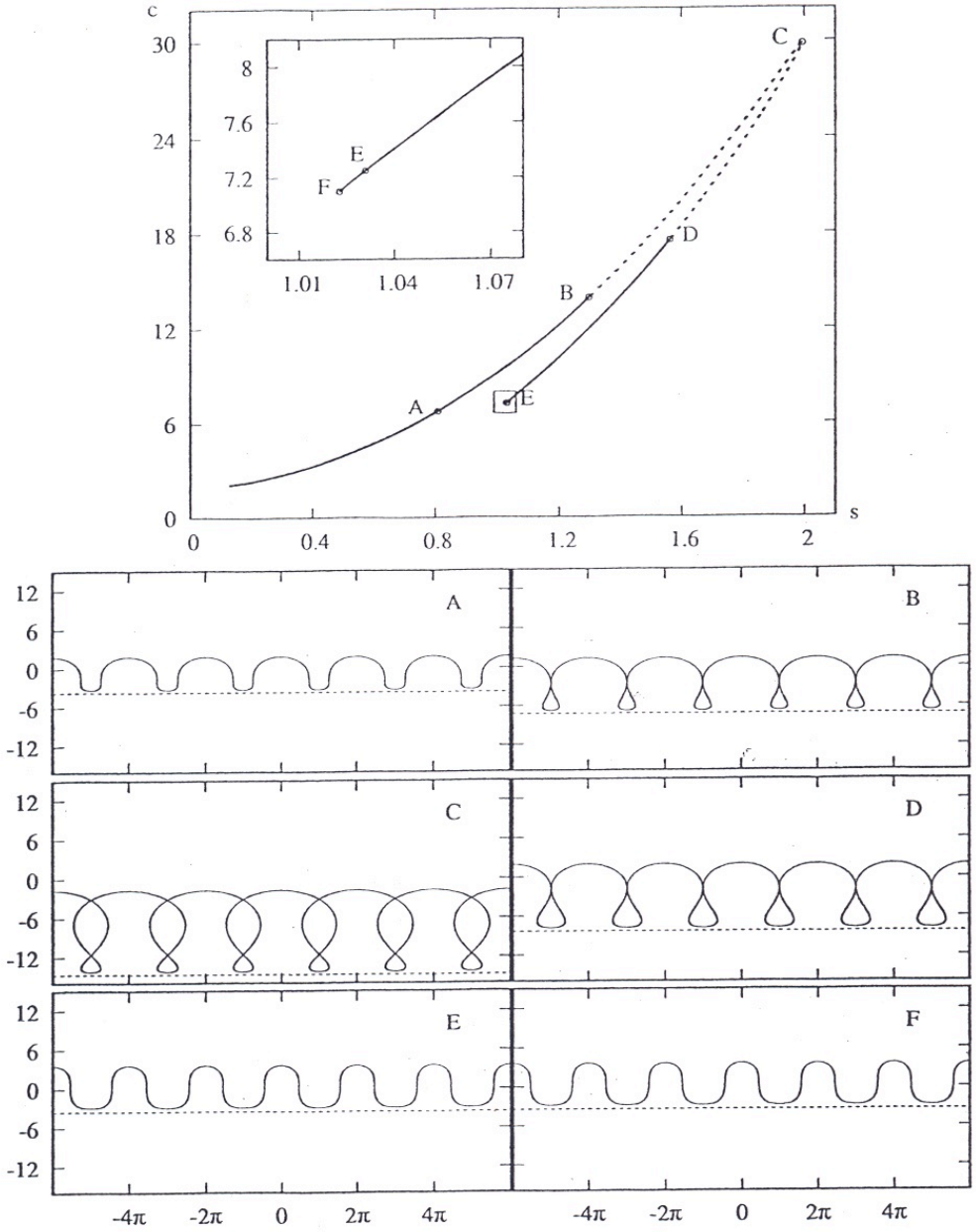


FIGURE 4. (top) Wave speed versus steepness for $\omega = 2$ and $d = 1$. The inset is a closeup near the end point of the numerical continuation. (bottom) Wave profiles of the solutions labelled by A through F. The mean fluid surface is at $y = 0$ and the mean fluid depths are marked by dashed lines.

Extending the bifurcation curve beyond stagnation

For constant ω, we can eliminate the requirement that $u < c$, That is, we can permit critical and stagnation points on the wave, even at the beginning of the bifurcation. Thus we can extend the curve $\mathcal{C}$ beyond such points.
For that reason we can also eliminate the assumption that "either $\gamma \leq 0$ or $|m|$ not too big or L is not too small".

Theorem 2. [A. Constantin, E. Varvaruca &S (2016)]
Let the vorticity ω be a constant and also fix $L > 0$ and $h > 0$. We let m and Q vary. Then there is a local curve $\mathcal{K}_{loc}$ of nontrivial solutions bifurcating from a trivial solution.

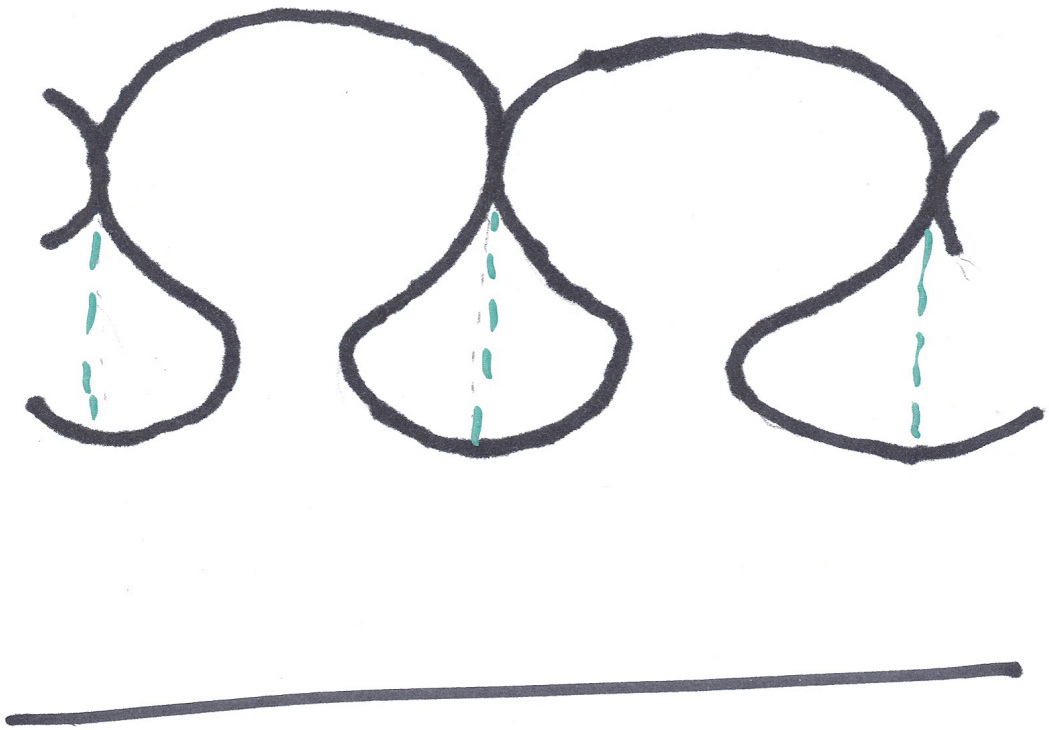
The bifurcation curve can be globally continued in the following sense. There exists a continuous curve $\mathcal{K}$ of such C^∞ solutions that continues the local curve such that one of the following alternatives is valid.

Three alternatives for $\mathcal{K}$

- either $\mathcal{K}$ is unbounded in $\mathbb{R} \times \mathbb{R} \times C^{1+\alpha}_{per}(\mathbb{R})$
- or $\mathcal{K}$ approaches self-intersection directly above the trough
- or the waves in $\mathcal{K}$ approach their greatest possible height; that is, $Q - 2gY \searrow 0$ for some sequence along $\mathcal{K}$.

Each solution corresponds to a water wave.
Each solution has a single crest and trough within its period L, and the amplitude Y is strictly monotone between them.
(The choice of variables that we use is explained later.)

The Dyachenko-Hur computations illustrate the self-intersection case.



The favorable waves

“Favorable" or "downstream" means $\omega < 0$ (given the appropriate choice of bifurcation). For this branch of $\mathcal{K} = \mathcal{K}(\omega)$, two of the three alternatives cannot happen. The intuition is that a favorable wind will lead to waves that break more easily. The breaking, which leads to time-dependent waves, will happen at smaller amplitudes.

Theorem 3. [A. Constantin, E. Varvaruca &S (2017)]
If constant $\omega < 0$, then
(i) The waves do not overhang (no stagnation points).
(ii) The wave surface S is real-analytic.
(iii) For fixed ω, the waves approach their maximum possible amplitude $Q/2g$, as one moves along the curve $\mathcal{K}(\omega)$.
(iv) As ω varies, the maximum amplitude along all the curves $\mathcal{K}(\omega)$ is bounded by a constant independent of ω.
(v) The maximum amplitude on $\mathcal{K}(\omega)$ tends to zero as $\omega \to -\infty$.

Steepness

Theorem 4. [Miles Wheeler & S (2016)]
For favorable vorticity and at least until $|(u-c)\omega| < g$, the maximum angle of inclination θ of all the waves in $\mathcal{K}$ is $< 45°$.

Numerics indicate that the condition $|(u-c)\omega| < g$ is true along all of $\mathcal{K}(\omega)$. [Katie Oliveras (2018)]

Similar results are true for small adverse vorticity, $0 < \omega < \delta(L, m)$. [Seung Wook So & S (2017)]

Fixing the boundary #1

First step in both the proof of Theorem 1 and the simulations: fix the free boundary by introducing a new independent variable

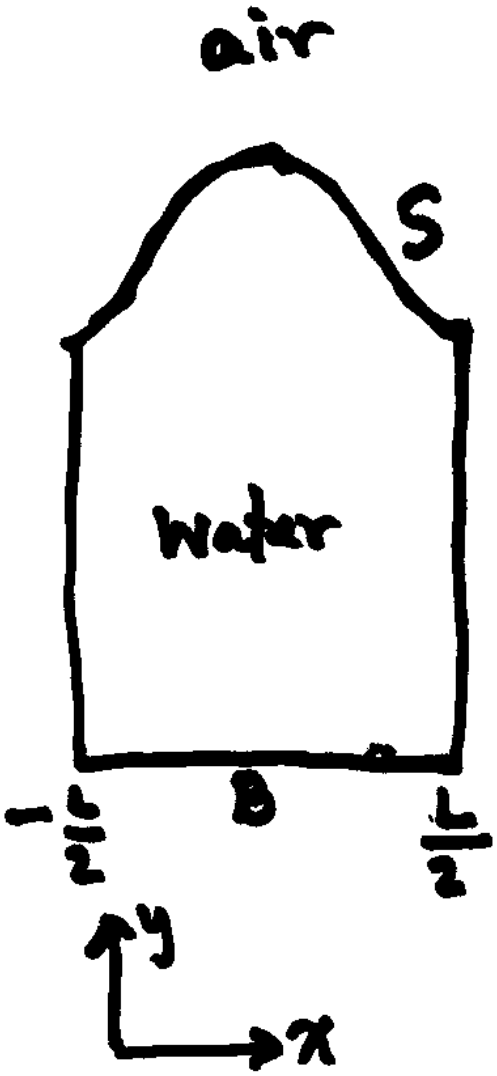
$$p = cy - \psi(x, y)$$

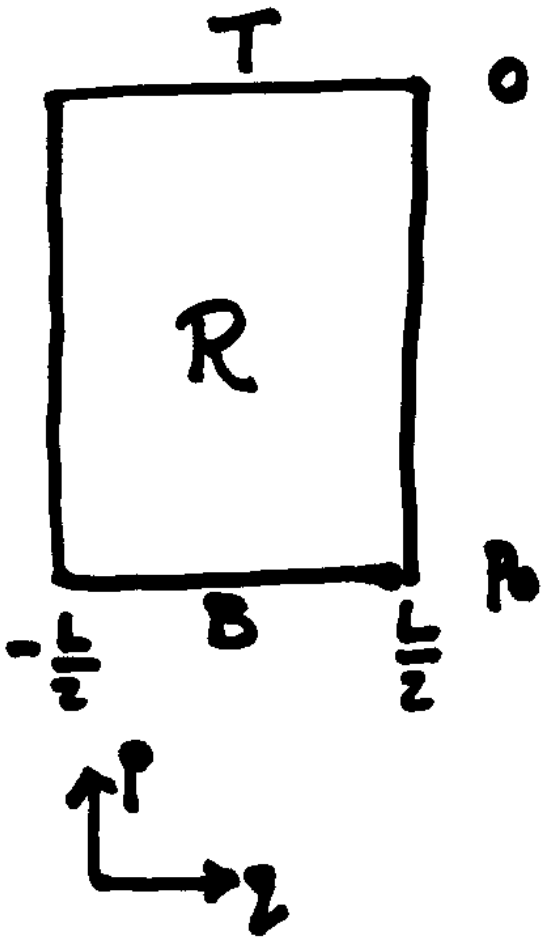
.

(It would be very nice for the other coordinate q to be its harmonic conjugate but that is possible only if ψ is harmonic, that is, if there is no vorticity.)

So Dubreil-Jacotin in 1933 simply put $q = x$. This permits arbitrary vorticity.

D-J Transformation:





The D-J transformation

$$p = cy - \psi(x, y), \quad q = x$$

leads to the following system for the height $h(q, p) = y + d$ in the rectangle $R = \{0 < q < L,\ p_0 < p < 0\}$:

$$\left\{ -\frac{1 + h_q^2}{2h_p^2} + \Gamma(p) \right\}_p + \left\{ \frac{h_q}{h_p} \right\}_q = 0 \quad \text{in} \quad R,$$

$$-\frac{1 + h_q^2}{2h_p^2} - gh + \frac{Q}{2} = 0 \quad \text{on the top} \quad T$$

$$h = 0 \quad \text{on the bottom } B$$

period L in the q variable.

Notice that the PDE is elliptic so long as $h_p > 0$.
Because $h_p = 1/(c - u)$, this means we will require $u < c$.
Notice that the only free parameters are the head Q, the dimensions of R, and the vorticity function $\gamma = \Gamma'$.

Fixing the boundary #2, for constant vorticity

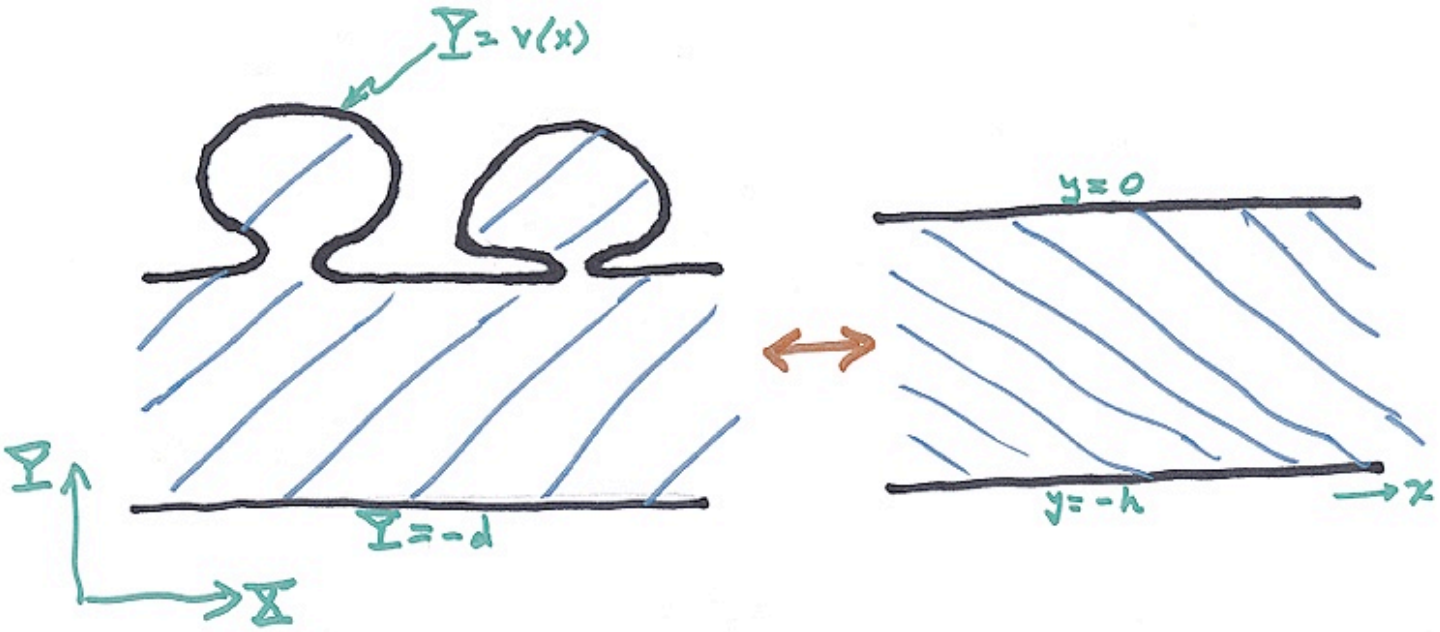
For constant vorticity we derive a new formulation as follows.

- Subtract $\frac{\omega}{2}Y^2$ to get a harmonic function.
- Conformally map $(X, Y) \to (x, y)$ from the fluid domain Ω onto the infinite strip $\{-\infty < x < \infty,\ -h < y < 0\}$.
- Use the Green's function (Dirichlet-Neumann operator) of the strip to reduce to an equation on the top surface T of the strip.

(We also obtain an equivalent variational characterization of the problem by converting the energy

$$E = \iint \{\tfrac{1}{2}|\nabla\psi|^2 - \omega\psi + \tfrac{Q}{2} - gY\} dY dX$$

to the new coordinates.)



Conformal Map

Finite Hilbert transform

Take period $L = 2\pi$. The Dirichlet-Neumann operator $\mathcal{G}$ on the strip of width h (= the "conformal depth") is given by

$$\mathcal{G}w = \mathcal{C}w_x + \frac{1}{h}[w]$$

where we denote $[\ \ ]$ = average over a period and where $\mathcal{C}$ is the finite Hilbert transform. $\mathcal{C}$ is defined as follows. Let w be a 2π-periodic function with average zero:

$$w(x) = \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Then

$$\mathcal{C}w(x) = \sum_{n=1}^{\infty}(a_n \coth nh\ \cos nx + b_n \coth nh\ \sin nx).$$

It reduces to the usual Hilbert transform as $h \to \infty$.

Transformed equations

The unknown $w(x)$ in the new formulation is the height Y of the surface S expressed in the new coordinate x. So it is a function of a single variable. The formulation is as follows (after a *long* calculation):

$$\begin{aligned}\mathcal{C}\{(Q-2gw-\omega^2w^2)w_x\}+(Q-2gw-\omega^2w^2)\{1+\mathcal{C}w_x\}\\ +\tfrac{2\omega m}{h}w+\tfrac{\omega^2}{h}[w^2]w+2\omega^2\mathcal{C}(ww_x)w\\ -(Q+2\omega m-2gh)+2g[w\mathcal{C}w_x]=0\end{aligned} \tag{1}$$

(with period 2π), together with the scalar equation

$$\begin{aligned}&\left[\left\{m/h+\omega[w^2]/2h+\omega\mathcal{C}(ww_x)-\omega w-\omega w\mathcal{C}w_x\right\}^2\right]\\ &=\left[(Q-2gw)\left\{w_x^2+1+2\mathcal{C}w_x+(\mathcal{C}w_x)^2\right\}\right].\end{aligned} \tag{2}$$

We fix $g>0, h>0$ and $\omega\in\mathbb{R}$ and look for solutions $(Q,m,w)\in\mathbb{R}\times\mathbb{R}\times\aleph$ of this system, where $\aleph$ is a Hölder space of smooth 2π-periodic functions on $\mathbb{R}$. [ ] = average.

Basic Structure

The main equation (5) has the structure

$$2(Q - 2gw)\mathcal{C}w_x + J(w) = 0 \tag{3}$$

with $\mathcal{C}$ of order zero. In terms of w, the first term has order one and $J(w)$ is the sum of many terms of order zero. For example,

$$\mathcal{C}\{(Q - 2gw)w_x\} - (Q - 2gw)\mathcal{C}w_x \tag{4}$$

has order zero. Thus the main equation is elliptic of order one on the circle provided $Q - 2gw \neq 0$.

Equation (6) has two roots for m, corresponding to the two branches of $\mathcal{K}$ coming from the two different bifurcation points.

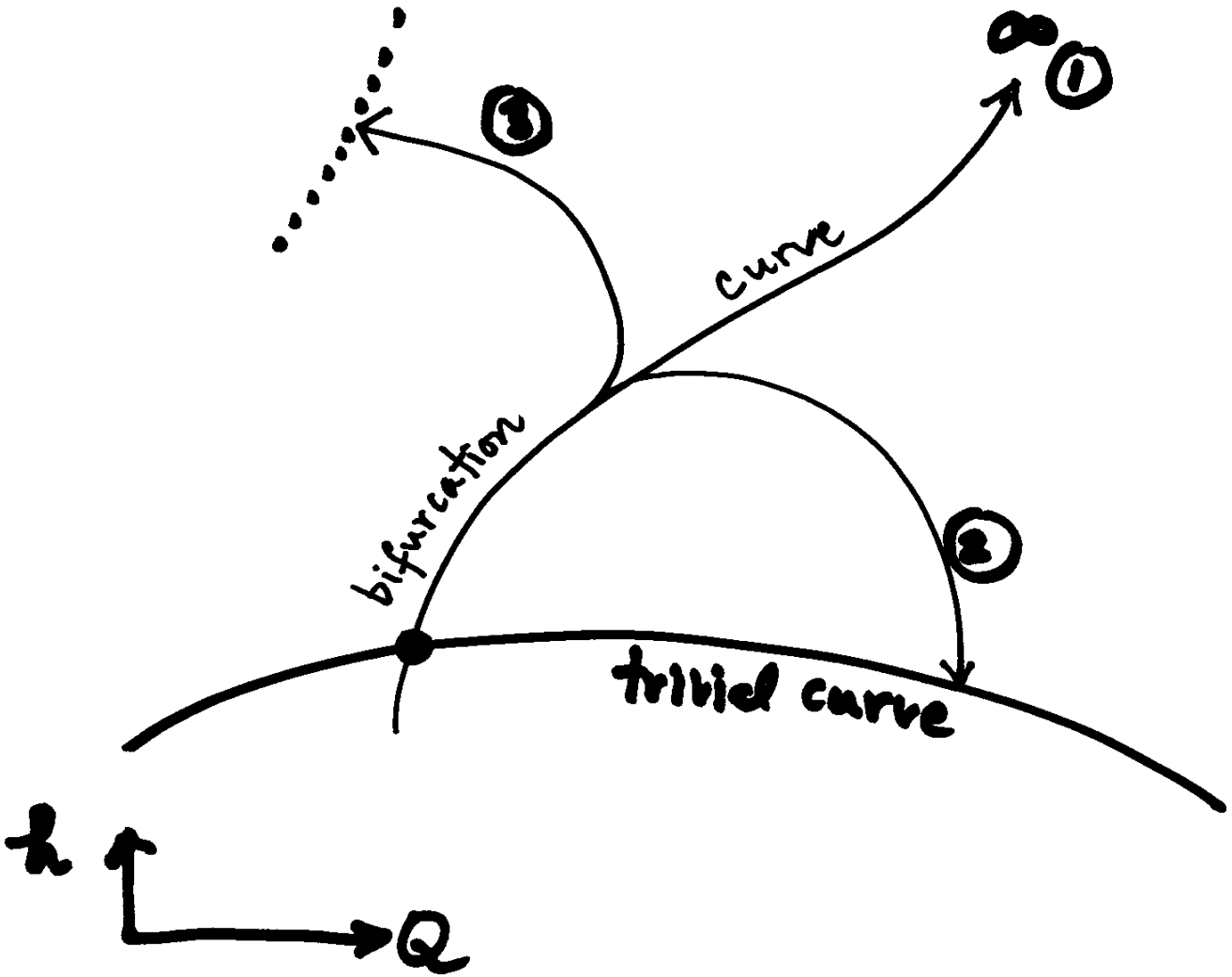
Proof of Theorem 1

First, local bifurcation.
We treat Q as a bifurcation parameter.
Suppose $\gamma(\cdot)$ is an arbitrary smooth function.

There is a curve of trivial (flat surface) solutions $Q(\lambda), H(p;\lambda)$ in the (Q, h) space $\mathbb{R} \times C^{3,\alpha}$, where $0 < \alpha < 1$.

We look along this curve for a bifurcation point $Q(\lambda^*), H(p, \lambda^*)$.
At such a point the linearized operator must have a kernel.

Even though this is not a standard eigenvalue problem, it is still possible to apply the Crandall-Rabinowitz bifurcation - from - a - simple - eigenvalue theorem (1971) (or the Liapunov-Schmidt method) to get a unique local curve of non-trivial solutions.
Their amplitudes are small.


∞
1
2
3
curve
bifurcation
trivial curve
h
Q

Global Continuation

We want to continue the bifurcation curve in $\mathbb{R} \times C^{3,\alpha}$.
Our tools are:

- Degree theory, Leray & Schauder (1930's), but specifically the degree of Healey & Simpson (1998).
- Global bifurcation method of Rabinowitz (1971).
- Schauder estimates for elliptic operators.

Alternative tool: the Dancer-Buffoni-Toland analytic theory.

We obtain one of the following possibilities:

1. Unboundedness: $\mathcal{C}$ becomes unbounded.
2. Self-intersection: $\mathcal{C}$ intersects the trivial curve at another point.
3. Degeneracy: The PDE becomes non-elliptic or the BC becomes non-oblique.

Degree

The degree is like the winding number but here it is in infinite-dimensional function space.

Let

$$\mathcal{G} : \Omega \to Y, \quad \Omega \subset X,$$

such that $\mathcal{G}^{-1}$ maps compact sets into compact sets.
For “regular" values, y,

$$deg(\mathcal{G}, \Omega, y) = \sum_{\mathcal{G}(w)=y,\ w\in\Omega} (-1)^{\nu(w)}$$

where $\nu(w)$ is the number of positive real eigenvalues of the linearized operator $\mathcal{G}'(w)$.

- Eliminate self-intersection by using the nodal properties (nonvanishing derivative between crest and trough).
  Tool: the maximum principles of Hopf and Serrin (1971).

- Reduce $\|h\|_{C^{3,\alpha}} \to \infty$ (unboundedness of h) to the much weaker statement that $\|h_p\|_{L^\infty} \to \infty$ along $\mathcal{C}$.
  Tool: the Lieberman-Trudinger regularity estimates (1986).

- Eventually reduce all possibilities to

$$\sup u \nearrow c \qquad \text{along } \mathcal{C}.$$

  Recall that $h_p = 1/(c-u)$.

  Tools: (1) A lower bound on u at the crest.
  (2) A lower bound on the pressure P.

Additional results on steady waves with vorticity

- Infinite depth. [Vera Hur]
- Stratified fluid. [Sam Walsh, Ming Chen]
- Solitary waves. [Wheeler]
- Surface tension: There can be multiple bifurcating curves even from the lowest eigenvalue. These curves are global. [Wahlén, Walsh]
- Vortex sheets. [Ambrose,. . .]
- Overturning internal waves. [Ambrose, Wright & S]
- Localized pressure disturbance on surface. [Wheeler]

250 years after Euler, still many fundamental open problems

- Criterion for overturning (breaking) waves
- Pressure beneath a steady wave
- 3D steady waves
  [Iooss and Plotnikov - bifurcation from hexagonal waves]
- Time-dependent flows:
- Unstable? Stable?
  [Zhiwu Lin - instability of some large amplitude waves]
- Roll-ups. Onset of turbulence. Fully turbulent flows.

Moral

The combination of

- careful modeling and physical intuition
- rigorous theory
- numerical simulations

is much more powerful than any one of the three alone!